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SKEW POLYNOMIAL RINGS OVER GENERALIZED GCD DOMAINS

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Abstract. A ring R is said to be a **right generalized GCD domain** if any finitely generated right v -ideal is a projective generator of the category $\text{Mod-}R$ of right R -modules. A skew polynomial ring $D[x, \sigma]$ over a commutative generalized GCD domain D is a right generalized GCD domain, where σ is an automorphism with finite order.

1 Preliminaries

At first, we introduce some elementary notions and notations. We refer to [MR] and [MMU] for details about orders and v -ideals.

Throughout this note, let R be an order in a division ring Q , that is, any non-zero element of R has its inverse in Q , and for any element q of Q , there exist $a, b \in R$ and non-zero $s, t \in R$ such that $q = as^{-1} = t^{-1}b$.

A non-zero right R -submodule I of Q is called a **right R -ideal** if there exists a non-zero element a of Q such that $aI \subseteq R$. Similarly, a **left R -ideal** of Q is a non-zero left R -submodule J of Q with $Ja \subseteq R$ for some non-zero element a of Q .

For any subsets A and B of Q , let

$$(A : B)_l = \{q \in Q \mid qB \subseteq A\}$$

and

$$(A : B)_r = \{q \in Q \mid Bq \subseteq A\}.$$

If I is a right R -ideal of Q , then $(R : I)_l$ is a left R -ideal. $(R : I)_r$ is a right R -ideal if J is a left R -ideal of Q .

For a right R -ideal I of Q , we set

$$I_v = (R : (R : I)_l)_r.$$

Clearly we have $I \subseteq I_v$, and I is called a **right v -ideal** if $I = I_v$. Furthermore, a right R -ideal I is said to be a **finitely generated v -ideal** if there exist finitely many elements

$a_1, \dots, a_k (\in I)$ such that $I = (a_1R + \dots + a_kR)_v$. Similarly, we set

$${}_vJ = (R : (R : J)_r)_l$$

for a left R -ideal J of Q . J is called a **left v-ideal** if $J = {}_vJ$, and J is said to be a **finitely generated left v-ideal** if $J = {}_v(Ra_1 + \dots + Ra_k)$ for some finitely many elements a_1, \dots, a_k of J .

For a right R -ideal I of Q , we put

$$O_r(I) = (I : I)_r = \{q \in Q \mid Iq \subseteq I\}.$$

$O_r(I)$ is called the **right order** of I . In fact, $O_r(I)$ is an order in Q . We define similarly the **left order** $O_l(I)$ of I :

$$O_l(I) = (I : I)_l = \{q \in Q \mid qI \subseteq I\},$$

and $O_l(I)$ is also an order in Q .

A right R -module M is called a **generator** of the category $\text{Mod-}R$ of right R -modules if $\sum_{f \in \text{Hom}_R(M, R)} f(M) = R$. We note that, for a right R -ideal I of Q , I is a generator of $\text{Mod-}R$ if and only if $(R : I)_l = R$. Furthermore, if I is a generator of $\text{Mod-}R$, then $O_l(I) = R$ (cf. Lemma 1.4 of [MMU]).

A right R -module M is said to be a **progenerator** of $\text{Mod-}R$ if M is a finitely generated projective R -module and a generator. Note that a right R -ideal I of Q is projective if and only if $I(R : I)_l = O_l(I)$. If I is projective, then I is finitely generated as a right R -module and $I_v = I$ (cf. Lemma 1.5 of [MMU]).

2 Right generalized GCD domains

A commutative domain is called a GCD domain if any non-zero two elements have the greatest common divisor. In a commutative domain D , the greatest common divisor d of elements a and b is characterized to be the element such that

$$dD = \bigcap_{eD \supseteq aD + bD} eD.$$

By Proposition 1.8 of [MMU], we have

$$\bigcap_{eD \supseteq aD + bD} eD = (aD + bD)_v.$$

Hence d is the greatest common divisor of a and b if and only if $dD = (aD + bD)_v$. Thus a domain is GCD if and only if any finitely generated v-ideal is principal.

Now, a principal ideal dD is clearly an invertible ideal, that is, $(dD)(dD)^{-1} = D$, where $(dD)^{-1} = \{q \in F \mid q(dD) \subseteq D\}$ and F is the quotient field of D . So, the notion

of a GCD domain is naturally extended to that of a generalized GCD domain, that is, a commutative domain D is called a generalized GCD domain if any finitely generated v -ideal of D is invertible (cf. [FHP] Chapter VI).

By the way, the polynomial ring $D[x]$ over a generalized GCD domain D is also a generalized GCD domain (cf. Theorem 6.2.3 of [FHP]). Then, what is a skew polynomial ring over a generalized GCD domain, or what is an Ore extension over a generalized GCD domain?

From these point of view, we define a non-commutative generalized GCD domain as follows: Let R be an order in a division ring Q . If any finitely generated right v -ideal of Q is a progenerator of $\text{Mod-}R$, then we call R a **right generalised GCD domain** (a **right G-GCD domain** for short), that is, R is G-GCD if

1. $(R : I)_t I = R$, and
2. $I(R : I)_t = O_t(I)$.

for any finitely generated right v -ideal I of Q . We note that a right Püfer order in Q is a right G-GCD domain (cf. [MMU]).

Now we have the following characterization of right G-GCD domains.

Theorem 2.1 *Let R be an order in a division ring Q . Then the following are equivalent:*

- (1) R is a right G-GCD domain.
- (2) For any non-zero elements a_1 and a_2 of R , the left R -ideal $Ra_1 \cap Ra_2$ is a progenerator of the category $R\text{-Mod}$ of left R -module.
- (3) For any left R -ideals J_1 and J_2 which are progenerator of $R\text{-Mod}$, $J_1 \cap J_2$ is also a progenerator of $R\text{-Mod}$.

3 Skew polynomial rings over generalized GCD domains

Let D be a commutative domain and let σ be an automorphism of D . Then we can define the skew polynomial ring $D[x, \sigma]$ over D with multiplication $xa = \sigma(a)x$, where $a \in D$. Since $D[x, \sigma]$ is a prime Goldie ring, $D[x, \sigma]$ has the quotient division ring Q .

We say that an automorphism σ of D has a **finite order** if $\sigma^k = \text{id}_D$ for some positive integer k , where id_D is the identity mapping of D .

Then we have the following.

Theorem 3.1 *Let D be a commutative generalized GCD domain and let σ be an automorphism of D with finite order. Then the skew polynomial ring $D[x, \sigma]$ is a right G-GCD domain.*

In particular, by Theorem 3.1, a skew polynomial ring over a commutative Prüfer domain is a right G-GCD domain. We note that the case of automorphisms with infinite order is an open problem. Also we don't know whether an Ore extension of a G-GCD domain is right G-GCD or not.

References

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